THE EXACT CONVERGENCE RATE IN THE ERGODIC THEOREM OF LUBOTZKY-PHILLIPS-SARNAK

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Abstract. We improve the upper bound of Lubotzky-Phillips-Sarnak on the discrepancy of points in the two-dimensional round sphere defined as orbits of Lipschitz quaternions. Rather than working with an explicit expression for the Hecke element defined by the characteristic function of a ball or a sphere of radius n in a finitely generated free group, we apply the spectral theorem and show that the deep and well-known inclusion of the spectrum of the Koopman operator defined by an Hecke element in the spectrum of the corresponding operator in the regular representation, leads to an upper bound on the discrepancy, which matches *exactly* the general lower bound of Shalom and Dudko-Grigorchuk.

Contents

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1. Introduction

1.1. **Convergence rates on the sphere.** In [28] and [29] the theory of automorphic forms and the theory of unitary representations are applied to compute the discrepancy of orbit points of Lipschitz quaternions on the 2-sphere. More precisely, let $q = x_0 + x_1 i + x_2 j + x_3 k$ be a quaternion and let $N(q) = x_0^2 + x_1^2 + x_2^2 + x_3^2$ be its norm. let p be a prime such that $p \equiv 1 \mod 4$. Let $\Sigma_{p+1} \subset \mathrm{SO}(3, \mathbb{R})$ denote the image of the set of Lipschitz quaternions

$$\{q = x_0 + x_1 i + x_2 j + x_3 k : x_0, x_1, x_2, x_3 \in \mathbb{Z} : N(q) = p : x_0 > 0 : x_0 \equiv 1 \mod 2\}$$

under the adjoint representation. Let ν be the probability Lebesgue measure on the round sphere $\mathbb{S}^2$.

Theorem 1.1. *(Lubotzky-Phillips-Sarnak [28, Theorem 1.3, Theorem 1.5].) The subgroup Γ of $\mathrm{SO}(3, \mathbb{R})$ generated by the symmetric set Σ_{p+1} is free of rank $\frac{p+1}{2}$ and*

$$\sup_{\|f\|_2=1} \left\| x \mapsto \left(\frac{1}{|\Sigma_{p+1}|} \sum_{\gamma \in \Sigma_{p+1}} f(\gamma x) - \int_{\mathbb{S}^2} f(y) d\nu(y) \right) \right\|_2 = \frac{2\sqrt{p}}{p+1}.$$

Let E_n be either the sphere or the ball of Γ around $e \in \Gamma$ of radius n with respect to the word metric defined by Σ_{p+1}. There is a constant $C > 0$ such that for all $n \in \mathbb{N}$,

$$\sup_{\|f\|_2=1} \left\| x \mapsto \left(\frac{1}{|E_n|} \sum_{\gamma \in E_n} f(\gamma x) - \int_{\mathbb{S}^2} f(y) d\nu(y) \right) \right\|_2 \leq C n p^{-n/2}.$$

In the above statement, the expression $\|x \mapsto \varphi(x)\|_2$ denotes the L^2-norm $\|\varphi\|_2$ of a function φ. In what follow we will often shorten the notation, writing $\|\varphi(x)\|_2$ instead of $\|x \mapsto \varphi(x)\|_2$.

The next theorem is our main result. It generalizes [28, Theorem 1.3] and strengthens [28, Theorem 1.5].

Theorem 1.2. *Let Γ be the free subgroup of rank $\frac{p+1}{2}$ of $\mathrm{SO}(3, \mathbb{R})$ generated by the symmetric set Σ_{p+1}. Let S_n, respectively B_n, be the sphere, respectively the ball, of Γ around $e \in \Gamma$ of radius n with respect to the word metric defined by Σ_{p+1}. Then*

$$\sup_{\|f\|_2=1} \left\| \frac{1}{|S_n|} \sum_{\gamma \in S_n} f(\gamma x) - \int_{\mathbb{S}^2} f(y) d\nu(y) \right\|_2 = \left(1 + \frac{p-1}{p+1} n\right) p^{-n/2},$$

$$\sup_{\|f\|_2=1} \left\| \frac{1}{|B_n|} \sum_{\gamma \in B_n} f(\gamma x) - \int_{\mathbb{S}^2} f(y) d\nu(y) \right\|_2 = c(p,n) \left(1 + \left(1 + \frac{1}{\sqrt{p}}\right) n\right) p^{-n/2},$$

where $c(p,n) = \frac{p-1}{p+1-\frac{2}{p^n}}$.

The result is proved by establishing matching upper and lower bounds on the discrepancy (for the definition of the discrepancy, see Formula 1 below).

1.2. **Upper bounds: Weil and Deligne.** The upper bounds are obtained with the help of three main ingredients. The first ingredient is the inclusion of the spectrum of a Koopman operator, associated to the free subgroup of $\mathrm{SO}(3,\mathbb{R})$ generated by Lipschitz quaternions, and defined by an Hecke element, in the spectrum of the corresponding operator in the regular representation. See Formula 2 below for the precise statement. To the best of our knowledge, the only known proof of this inclusion, is the one from [28, S153-S158] and [29, Theorem 4.1], which uses the theory of automorphic forms and Deligne's solution to the Weil conjecture. The second ingredient is an application of the spectral theorem to Hecke elements. The third one is an identity between the norm of operators defined by the regular representation of a free group and values of the Harish-Chandra function of a free group: see Proposition 4.2 below.

1.3. **Lower bounds: a general fact.** The lower bounds follow from a general result of Shalom announced in [35, Theorem 4.14] and also stated in [17, Proposition 7]: if a finitely generated group Γ acts by measure-preserving transformations on an atomless probability space (X,ν), then there is a subgroup H of Γ, such that the quasi-regular representation of Γ on $l^2(\Gamma/H)$, is weakly contained in the restriction of the Koopman representation of Γ to the orthogonal complement $L^2_0(X,\nu)$ of the constant functions. In the following proposition, we spell-out the consequence of this result we need.

Proposition 1.3. *Assume that Γ is a free group of rank $r \geq 1$ acting by measure-preserving transformations on an atomless probability space (X,ν). Let $\{a_1,\dots,a_r\}$ be a free generating set of Γ. Let $S=\{a_1^{\pm 1},\dots,a_r^{\pm 1}\}$. Let $q=2r-1$. Let S_n, respectively B_n, be the sphere, respectively the ball, of Γ around $e\in\Gamma$ of radius n with respect to the word metric defined by S. Then*

$$\sup_{\|f\|_2=1}\left\|\frac{1}{|S_n|}\sum_{\gamma\in S_n} f(\gamma x)-\int_X f(y)d\nu(y)\right\|_2 \geq \left(1+\frac{q-1}{q+1}n\right)q^{-n/2},$$

$$\sup_{\|f\|_2=1}\left\|\frac{1}{|B_n|}\sum_{\gamma\in B_n} f(\gamma x)-\int_X f(y)d\nu(y)\right\|_2 \geq c(q,n)\left(1+\left(1+\frac{1}{\sqrt{q}}\right)n\right)q^{-n/2},$$

where $c(q,n)=\left(1+2q^{-n}\sum_{k=0}^{n-1}q^k\right)^{-1}$.

Remark 1.4. *Notice that both lower bounds evaluated at $q=1$ give the value 1 and in this case both inequalities are equalities. When $q>1$, then $c(q,n)=\frac{q-1}{q+1-\frac{2}{q^n}}$.*

1.4. **Convergence rates on the torus.** It follows from [28, Theorem 1.4] that a generic finitely generated free subgroup of $\mathrm{SO}(3,\mathbb{R})$ does *not* realize the lower bounds of Proposition 1.3. This is in contrast with the group of automorphisms of the torus where any finitely generated free subgroup realizes the fastest possible convergence rate, as stated in the following theorem which easily follows from [35, Theorem 4.17] or [16, 20], or ideas presented in [19], or [20].

Theorem 1.5. *Let $\mathbb{T}^2=\mathbb{R}^2/\mathbb{Z}^2$ be the 2-torus with its normalized Haar measure ν and let $GL(2,\mathbb{Z})\simeq Aut(\mathbb{T}^2)=G$ be its automorphism group. Assume that $\Gamma<G$ is a free subgroup of rank $r\geq 1$ freely generated by $\{a_1,\dots,a_r\}\subset G$. Let $S=\{a_1^{\pm 1},\dots,a_r^{\pm 1}\}$. Let $q=2r-1$. Let S_n, respectively B_n, be the sphere, respectively*

the ball, of Γ *around* $e \in \Gamma$ *of radius* n *with respect to the word metric defined by* S*. Then*

$$\sup_{\|f\|_2=1} \left\| \frac{1}{|S_n|} \sum_{\gamma \in S_n} f(\gamma x) - \int_{\mathbb{T}^2} f(y) d\nu(y) \right\|_2 = \left(1 + \frac{q-1}{q+1} n\right) q^{-n/2},$$

$$\sup_{\|f\|_2=1} \left\| \frac{1}{|B_n|} \sum_{\gamma \in B_n} f(\gamma x) - \int_{\mathbb{T}^2} f(y) d\nu(y) \right\|_2 = c(q,n) \left(1 + \left(1 + \frac{1}{\sqrt{q}}\right) n\right) q^{-n/2},$$

where $c(q,n) = \left(1 + 2q^{-n} \sum_{k=0}^{n-1} q^k\right)^{-1}$.

1.5. **Related references.** We close this introduction by mentioning several works related to [28] and [29]. The five pages paper of Arnol'd and Krylov [1] is one of the earlier reference on free groups of rotations. Lubotzky's book [27] (specially Chapter 9) is a general reference to the subject and its numerous ramifications. Colin de Verdière has given a séminaire Bourbaki [13] on [28], [29]. Shalom's survey [35] presents estimates of the discrepancy of random points. In a series of paper, Bourgain and Gamburd (see [5] and references therein) construct finite symmetric sets in $\mathrm{SU}(d)$ whose Koopman's operators have norms smaller than 1. Clozel [10] has obtained sharp bounds (up to multiplicative constants) on the discrepancies of some subsets of $\mathrm{SO}(2n)$. In [11], Clozel, Oh, and Ullmo, express convergence rates for ergodic theorems on locally symmetric spaces, in terms of Harish-Chandra functions. The survey [22] of Gorodnik and Nevo includes many convergence rates estimates. In [18], Ellenberg, Michel, Venkatesh, discuss and improve Linnik's work on the distribution of the spatial distribution of point sets on the 2-sphere, obtained from the representation of a large integer as a sum of three integer squares. In [6] and [7], Bourgain, Rudnick, Sarnak, evaluate this distribution through different "statistics", and compare it with what's happening in higher dimension, and with the case of random points. Parzanchevski and Sarnak have shown that the optimal generating rotations $\Sigma_{p+1} \subset \mathrm{SO}(3,\mathbb{R})$ can be used to construct efficient gates needed as building blocks for quantum algorithms [31].

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2. Discrepancy and the Koopman representation

In this section we recall the relevant definitions and notation from representation theory needed for the proofs of the results stated in the introduction.

2.1. **Three involutive algebras.** Let Γ be a group. Recall that the formal linear combinations of elements of Γ with complex coefficients

$$\sum_{\gamma \in \Gamma} a_\gamma \gamma,$$

form the group algebra $\mathbb{C}[\Gamma]$ of Γ over the complex numbers. Each element of $\mathbb{C}[\Gamma]$ has a norm defined as:

$$\left\| \sum_{\gamma \in \Gamma} a_\gamma \gamma \right\|_1 = \sum_{\gamma \in \Gamma} |a_\gamma|.$$

The space

$$l^1(\Gamma) = \{ f : \Gamma \to \mathbb{C} : \|f\|_1 = \sum_{\gamma \in \Gamma} |f(\gamma)| < \infty \}$$

of summable functions on Γ, is a convolution algebra for the law

$$f * g(x) = \sum_{\gamma \in \Gamma} g(\gamma^{-1} x) f(\gamma), \ \forall x \in \Gamma.$$

There is a unique embedding of involutive unital algebras of $\mathbb{C}[\Gamma]$ into $l^1(\Gamma)$, which sends each $\gamma \in \mathbb{C}[\Gamma]$ to the characteristic function $\delta_\gamma \in l^1(\Gamma)$ of γ. Let $\pi : \Gamma \to U(\mathcal{H})$ be a unitary representation of Γ on a Hilbert space $\mathcal{H}$. Let $B(\mathcal{H})$ be the involutive algebra of bounded operators on $\mathcal{H}$. If $T \in B(\mathcal{H})$ we denote by $\|T\|$ its operator norm. There is a unique morphism of involutive unital algebras from $l^1(\Gamma)$ to $B(\mathcal{H})$ whose restriction to Γ equals π. We also denote this morphism by π.

Let E be a finite subset of Γ and let $|E|$ be its cardinality. We denote the element of $\mathbb{C}[\Gamma]$ defined as the sum over elements of E by

$$\mathbf{1}_E = \sum_{\gamma \in E} \gamma$$

and we define μ_E to be $\frac{1}{|E|}\mathbf{1}_E$.

Let $l^2(\Gamma)$ be the Hilbert space of square integrable functions on Γ. Let $\rho_\Gamma : \Gamma \to U(l^2(\Gamma))$ be the right regular representation:

$$\rho_\Gamma(\gamma) f(x) = f(x\gamma), \ \forall x, \gamma \in \Gamma.$$

2.2. **Koopman representations.** Let X be a probability space and let ν be a probability measure on X without atom. Let $G = \mathrm{Aut}(X, \nu)$ be the group of all measure-preserving transformations of X. (Any element in G is represented by a map $g : X \to X$ which is one-to-one and onto, such that $\nu(g^{-1}B) = \nu(B)$ for any B in the σ-algebra on which ν is defined. Two such transformations are identified if and only if they coincide on a set of full measure.) Let $\mathcal{H} = L^2(X, \nu)$ be the Hilbert space of complex square integrable functions on X. If $f \in L^2(X, \nu)$ we denote its norm by $\|f\|_2$. Let $\pi : G \to U(L^2(X, \nu))$ be the Koopman representation:

$$\pi(g) f(x) = f(g^{-1} x).$$

2.3. **Discrepancy.** Let $1_X \in \mathcal{H}$ be the characteristic function of X. Let $P \in B(\mathcal{H})$ be the orthogonal projection onto the complex line generated by 1_X. We have $P^2 = P$, $P = P^*$, $P\pi(g) = \pi(g)P = P$ for all $g \in G$. Let π_0 denote the restriction of π to the kernel

$$\mathrm{Ker} P = L^2_0(X, \nu) = \{ f \in L^2(X, \nu) : \int_X f(x) d\nu(x) = 0 \}.$$

Definition 2.1. *Let $E = E^{-1}$ be a finite subset of G. The discrepancy of E is defined as the norm $\|\pi_0(\mu_E)\|$ of the operator $\pi_0(\mu_E) : L^2_0(X, \nu) \to L^2_0(X, \nu)$.*

It is easy to check that

$$\|\pi_0(\mu_E)\| = \sup_{\|f\|_2=1} \left\| \frac{1}{|E|} \sum_{\gamma\in E} f(\gamma x) - \int_X f(y) d\nu(y) \right\|_2 , \tag{1}$$

where the supremum is taken over f running in the unit sphere of the whole Hilbert space $L^2(X,\nu)$.

3. Proofs

To prove the results stated in the introduction, we start with a general lower bound on the discrepancy which is sharp in the case of the isometries of the sphere, as well as in the case of the automorphisms of the torus. We then prove the upper bounds in the case of the sphere. Finally we prove the upper bounds in the case of the torus.

3.1. **A lower bound for the discrepancy.**

We present two proofs of the following lower bound which generalizes and strengthens [34, Théorème 3.3], [10, Theorem 2], [32, Theorem 4], [28, Theorem 1.3]. The first proof is based on a stronger result due to Shalom [35, Theorem 4.14] which also appears in [17, Proposition 14]. The second proof we give is short and self-contained.

Theorem 3.1. *Let (X,ν) be an atomless probability space. Let Γ be a finitely generated group of measure-preserving transformations of X. Let $m\in\mathbb{C}[\Gamma]$ be a positive element (i.e. $m=\sum_{\gamma\in\Gamma} a_\gamma\gamma$ with $a_\gamma\geq 0$). Then:*

$$\|\pi_0(m)\| \geq \|\rho_\Gamma(m)\|.$$

Remark 3.2. *Let $\mathcal{H}$ be a Hilbert space and let $\pi:\Gamma\to U(\mathcal{H})$ be a unitary representation. The unique morphism $\pi: l^1(\Gamma)\to B(\mathcal{H})$ of unital involutive algebras extending π is continuous. (More precisely it satisfies $\|\pi(f)\|\leq\|f\|_1$ for all $f\in l^1(\Gamma)$.) As a consequence, Theorem 3.1 extends to positive elements in $l^1(\Gamma)$.*

Remark 3.3. *As mentioned to us by P.-E. Caprace, the existence of measure preserving actions, of locally compact* amenable *groups, with a spectral gap (see for example* [30, Corollary 1.10, 1, Chap. III]*), prevents an obvious generalization of the above result to locally compact groups. But, we believe the following statement is true. Let μ be a Haar measure on a locally compact group G. Let ρ_G be the right-regular representation of G. Let X be a Hausdorff topological space. Let ν be an atomless Borel regular probability measure on X. Suppose that G acts continuously on X, and preserves ν. Assume there exists a point x_0 in the support of ν, with compact stabilizer, such that the orbit of x_0, under any compact subset K of G, has zero measure: $\nu(Kx_0)=0$. Let $\pi_0: L^1(G,\mu)\to B(L^2_0(X,\nu))$ be the Koopman representation, restricted to the subspace of functions with zero integral. Then, for any continuous function $f:G\to\mathbb{R}$, with compact support, and taking only nonnegative values, we have:*

$$\|\pi_0(f)\| \geq \|\rho_G(f)\|.$$

And if G is is second countable, the inequality extends to nonnegative functions belonging to $L^1(G,\mu)$. (We hope to say more on these questions in a forthcoming paper.)

First proof of Theorem 3.1.

Proof. According to [35, Theorem 4.14] or [17, Proposition 7], there exists a subgroup H of Γ such that the quasi-regular representation of Γ on $l^2(\Gamma/H)$ is weakly contained in the restriction of the Koopman representation of Γ to the orthogonal complement $L_0^2(X,\nu)$ of the constant functions. As the quasi-regular representation of Γ on $l^2(\Gamma/H)$ contains positive vectors, the theorem follows from [36, Lemma 2.3], and the definition of weak containment [2, Definition F.1.1]. $\square$

Second proof of Theorem 3.1.

Proof. If $m=0$ the inequality is trivial. If $m \neq 0$, multiplying m by $\|m\|_1^{-1}$, we may assume that $\|m\|_1 = 1$. As $\|T\|^2 = \|TT^*\|$ for any bounded operator T on a Hilbert space, we may moreover assume $m = m^*$.

Let $e \in \Gamma$ be the identity element. We claim that for any $n \in \mathbb{N}$, $\|\pi_0(m^n)\| \geq m^{(n)}(e)$, where $m^{(n)} \in l^1(\Gamma)$ is the n-th convolution power of m (although m has finite support it is convenient here and in what follows to view m in the convolution algebra $l^1(\Gamma)$ of summable functions).

To prove this claim, let F be the support of $m^{(n)}$. We choose a measurable subset B_+ of X such that

$$0 < \nu(B_+) < \frac{1}{2|F|}.$$

Such a set obviously exists because ν is finite without atom. As the action preserves the measure,

$$\nu(FB_+) \leq |F|\nu(B_+) < 1/2.$$

A finite atomless measure has the intermediate value property (see [3, 1.12.10 Corollary]) hence there exists a measurable subset B_- of $X \setminus FB_+$ satisfying $\nu(B_+) = \nu(B_-)$. Let

$$\varphi = \frac{1_{B_+} - 1_{B_-}}{\|1_{B_+} - 1_{B_-}\|_2} \in L_0^2(X,\nu).$$

The proof of the claim follows then from the Cauchy-Schwarz inequality, the symmetry of F, and the positivity of m:

$$\begin{aligned}
\|\pi_0(m^n)\| &\geq \langle \pi_0(m^n)\varphi, \varphi\rangle \\
&= \frac{1}{\nu(B_+)+\nu(B_-)} \sum_{\gamma\in\Gamma} [\nu\left(\gamma B_+ \cap B_+\right) + \nu(\gamma B_- \cap B_-)] m^{(n)}(\gamma) \\
&\geq \frac{1}{\nu(B_+)+\nu(B_-)} \sum_{\gamma=e} [\nu\left(\gamma B_+ \cap B_+\right) + \nu(\gamma B_- \cap B_-)] m^{(n)}(\gamma) \\
&= m^{(n)}(e).
\end{aligned}$$

This finishes the proof of the claim.

Applying the claim we obtain:

$$\begin{aligned}
\|\pi_0(m)\| &= \lim_{n\to\infty} \|\pi_0(m)^n\|^{1/n} \\
&\geq \limsup_{n\to\infty} m^{(n)}(e)^{1/n} \\
&= \|\rho_\Gamma(m)\|.
\end{aligned}$$

The last equality above goes back to [24, Lemma 2.2]. $\square$

The following corollary together with Proposition 1.3 illustrate Theorem 3.1 with two opposite situations: the discrepancy is maximal in the case the transformations generate an amenable group whereas the discrepancy may be small in the case the transformations generate a free group.

Corollary 3.4. *Let $E \subset G$ be finite and symmetric. Let Γ be the group generated by E. If Γ is amenable then*

$$\sup_{\|f\|_2=1} \left\| \frac{1}{|E|} \sum_{\gamma \in E} f(\gamma x) - \int_X f(y) d\nu(y) \right\|_2 = 1.$$

Proof. Without any hypothesis on Γ we always have $\|\pi_0(\mu_E)\| \leq \|\mu_E\|_1 = 1$. According to Theorem 3.1, $\|\pi_0(\mu_E)\| \geq \|\rho_\Gamma(\mu_E)\|$. According to [25], the group Γ is amenable if and only if $\|\rho_\Gamma(\mu_E)\| = 1$. □

We prove Proposition 1.3.

Proof. Let $n \in \mathbb{N}$ be given. Let $S_n \subset \Gamma$ be the sphere around e of radius n. As explained in Proposition 4.2 from the Appendix, or according to [12] or [37, 12.17],

$$\|\rho_\Gamma(\mu_{S_n})\| = \left(1 + \frac{q-1}{q+1} n\right) q^{-n/2}.$$

Let $E = S_n$ and let $H < \Gamma$ be the subgroup generated by E. Let $\mu_E \in \mathbb{C}[H] \subset \mathbb{C}[\Gamma]$. Decomposing Γ into its right H-cosets, it is easy to check that ρ_Γ restricted to H decomposes as a direct sum of unitary representations, all unitary equivalent to ρ_H, and that consequently

$$\|\rho_\Gamma(\mu_E)\| = \|\rho_H(\mu_E)\|.$$

Applying Theorem 3.1 we obtain

$$\|\pi_0(\mu_E)\| \geq \|\rho_H(\mu_E)\|.$$

Applying Formula 1 finishes the proof of the corollary in the case $E = S_n$. The case of the ball of radius n is similar. □

3.2. **Exact convergence rate for some isometries of the sphere.** We recall the construction from [29] of free subgroups of isometries of the round sphere. Let $\mathbb{H} = \{q = x_0 + x_1 i + x_2 j + x_3 k : x_0, x_1, x_2, x_3 \in \mathbb{R}\}$ be the field of quaternions. Let

$$\tau(q) = \overline{x_0 + x_1 i + x_2 j + x_3 k} = x_0 - x_1 i - x_2 j - x_3 k$$

denote the conjugate of q. Let $N(q) = q\overline{q}$ be the norm of q and let $|q| = \sqrt{N(q)}$ be its module. The multiplicative group $\mathbb{H}^*$ acts on $\mathbb{H}$ by conjugation and if $q \in \mathbb{H}^*$ and $v \in \mathbb{H}$, then $|qvq^{-1}| = |v|$. As this action preserves the subspace $\operatorname{Im}\mathbb{H} = \{x_1 i + x_2 j + x_3 k : x_1, x_2, x_3 \in \mathbb{R}\}$, it defines a homomorphism

$$\operatorname{Ad} : \mathbb{H}^* \to \mathrm{SO}(3, \mathbb{R}),\ q \mapsto (v \mapsto qvq^{-1})$$

with values in the orientation preserving isometry group of the round sphere $\mathbb{S}^2$. The ring

$$\mathbb{H}(\mathbb{Z}) = \{q = x_0 + x_1 i + x_2 j + x_3 k : x_0, x_1, x_2, x_3 \in \mathbb{Z}\}$$

of Lipschitz quaternions has 8 units:

$$\mathbb{H}(\mathbb{Z})^\times = \{\pm 1, \pm i, \pm j \pm k\}.$$

Let $n \in \mathbb{N}$. According to Jacobi (see for example [9, p. 27] or [14, Theorem 2.4.1] for odd integers), the cardinality of the set of Lipschitz quaternion of norm n is

$$|N^{-1}(n) \cap \mathbb{H}(\mathbb{Z})| = 8 \sum_{4 \nmid d | n} d.$$

Hence, if $n = p$ is prime, the set $N^{-1}(p) \cap \mathbb{H}(\mathbb{Z})$ splits as the disjoint union of the $p+1$ orbits of the action of $\mathbb{H}(\mathbb{Z})^\times$. In the case $p \equiv 1 \mod 4$, it is easy to check that each orbit contains a unique quaternion $q = x_0 + x_1 i + x_2 j + x_3 k$ with $x_0 > 0$ and $x_0 \equiv 1 \mod 2$ and that the set

$$\{q = x_0 + x_1 i + x_2 j + x_3 k : x_0, x_1, x_2, x_3 \in \mathbb{Z},\, N(q) = p,\, x_0 > 0,\, x_0 \equiv 1 \mod 2\}$$

splits into $\frac{p+1}{2}$ orbits of the involution τ, each containing two elements. Let $\Sigma_{p+1} \subset \mathrm{SO}(3, \mathbb{R})$ denote the image of this set under the homomorphism Ad.

We are ready to prove Theorem 1.2.

Proof. It follows from [28, S153-S158] and [29, Theorem 4.1], that the spectrum of π_0 satisfies

$$\sigma\left(\pi_0\left(\mathbf{1}_{\Sigma_{p+1}}\right)\right) \subset [-2\sqrt{p}, 2\sqrt{p}]. \tag{2}$$

(In fact it is shown in [28, S153-S158] that $\sigma\left(\pi_0\left(\mathbf{1}_{\Sigma_{p+1}}\right)\right) = [-2\sqrt{p}, 2\sqrt{p}]$. We will "only" use the inclusion $\sigma(\pi_0(\mathbf{1}_{\Sigma_{p+1}})) \subset [-2\sqrt{p}, 2\sqrt{p}]$ but this is by far the hardest to prove; the main ingredient in its proof is the inequality [29, Theorem 4.1] which relies in particular on [15].) Applying Inclusion 2 and Theorem 3.1 we deduce the inequalities

$$2\sqrt{p} \geq \left\|\pi_0\left(\mathbf{1}_{\Sigma_{p+1}}\right)\right\| \geq \left\|\rho_\Gamma\left(\mathbf{1}_{\Sigma_{p+1}}\right)\right\|.$$

It then follows from Kesten's spectral characterization of free groups (see [28, S157] and [24]) that Γ is free of rank $r = \frac{p+1}{2}$ (and freely generated by any subset A of Σ_{p+1} containing $\frac{p+1}{2}$ elements and satisfying $A \cap A^{-1} = \emptyset$). On Γ we consider the word metric defined by Σ_{p+1}, and for each $n \in \mathbb{N} \cup \{0\}$, the Hecke element

$$T_n = \sum_{|\gamma| = n} \gamma \in \mathbb{C}[\Gamma].$$

We have: $T_0 = e$,

$$T_1 = \sum_{\gamma \in \Sigma_{p+1}} \gamma = \mathbf{1}_{\Sigma_{p+1}},$$

$$T_1 T_1 = T_2 + 2r T_0.$$

If $n \geq 2$ we have:

$$T_n T_1 = T_{n+1} + p T_{n-1}.$$

There is a unique morphism of unital rings, from the ring $\mathbb{Z}[X]$ of polynomials in one variable with integer coefficients, to $\mathbb{C}[\Gamma]$, sending X to T_1. The above recursion relations show that T_n is in the image of this morphism for any $n \geq 0$. In other words for each $n \geq 0$, there exists $P_n \in \mathbb{Z}[X]$ such that $T_n = P_n(T_1)$.

We first prove the theorem in the case of a sphere $S_n \subset \Gamma$. The lower bound on the discrepancy follows from Proposition 1.3. For the upper bound, applying the spectral theorem for bounded self-adjoint operators, Inclusion 2, and Kesten's computation [24] of the spectrum of the regular representation

$$\sigma\left(\rho_\Gamma(T_1)\right) = [-2\sqrt{p}, 2\sqrt{p}],$$

we deduce that

$$\begin{aligned}\|\pi_0(T_n)\| &= \|\pi_0(P_n(T_1))\| \\ &= \|P_n(\pi_0(T_1))\| \\ &= \sup_{\lambda\in\sigma(\pi_0(T_1))} |P_n(\lambda)| \\ &\leq \sup_{\lambda\in[-2\sqrt{p},2\sqrt{p}]} |P_n(\lambda)| \\ &= \sup_{\lambda\in\sigma(\rho_\Gamma(T_1))} |P_n(\lambda)| \\ &= \|\rho_\Gamma(T_n)\|.\end{aligned}$$

Applying Proposition 4.2 or [37, 12.17], we conclude that

$$\sup_{\|f\|_2=1}\left\|\frac{1}{|S_n|}\sum_{\gamma\in S_n} f(\gamma x) - \int_{\mathbb{S}^2} f(y)d\nu(y)\right\|_2 \leq \left(1+\frac{p-1}{p+1}n\right)p^{-n/2}.$$

The proof, in the case of a ball B_n, follows exactly the same lines. □

3.3. **Exact convergence rate for automorphisms of the torus.** We prove Theorem 1.5.

Proof. According to Proposition 1.3, the lower bounds on the discrepancies are true and the cases with $q = 1$ have already been discussed in Corollary 3.4 and Remark 1.4. As explained in [16, 20] or in [35, Theorem 4.17], the restriction of the Koopman representation defined by the action of G on $\mathbb{T}^2$

$$\pi_0 : G \to U(L^2_0(\mathbb{T}^2, \nu))$$

is weakly contained in the regular representation ρ_G. Hence, according to Theorem [16, Theorem 7], for any $m \in \mathbb{C}[G]$,

$$\|\pi_0(m)\| \leq \|\rho_G(m)\|.$$

Choosing $m = \mu_{E_n} \in \mathbb{C}[\Gamma] \subset \mathbb{C}[G]$, where E_n is either a sphere or a ball, we get (as explained in the proof of Proposition 1.3):

$$\|\rho_G(\mu_{E_n})\| = \|\rho_\Gamma(\mu_{E_n})\|.$$

Applying Proposition 4.2 or [37, 12.17] finishes the proof of the theorem. □

4. Appendix

The aim of this appendix is to recall well-known facts about the regular and quasi-regular representations of the automorphism group of a regular tree. Most relevant for this paper are explicit formulae for the norms of Markov operators, defined by the regular representation of the automorphism group of the tree. Although the formulae from Proposition 4.2 below follow from [12], or [37, 12.17] which is based on [37, 12.10] (see also [33] for a more general setting), it seems worthwhile to emphasize that these formulae can also be deduced and expressed with the help of the Harish-Chandra function of the quasi-regular representation of the automorphism group of an homogeneous tree. (Both approaches are equivalent; the modular functions of cocompact amenable subgroups in [33] correspond to Radon-Nikodym cocycles of quasi-regular representations.)

4.1. **The boundary of a tree and Busemann cocycles.** Let (X, d) be the regular tree of degree $q+1$ equipped with its geodesic path metric d for which each edge is isometric to the unit interval $[0,1] \subset \mathbb{R}$. Let x_0 be a vertex of X. Let ∂X be its boundary at infinity (we refer the reader to [4] for more details). Let $b \in \partial X$ and let $\beta : [0, \infty) \to X$ be a geodesic ray representing b. Let $x, y \in X$. Let

$$B_b(x, y) = \lim_{t \to \infty} [d(x, \beta(t)) - d(y, \beta(t))],$$

be the Busemann cocycle defined by $b \in \partial X$. Let $a, b \in \partial X$ and let $\alpha, \beta : [0, \infty) \to X$ be geodesic rays representing a and b. Their Gromov product relative to the base point x_0 is defined as

$$(a|b)_{x_0} = \frac{1}{2} \lim_{t \to \infty} [d(x_0, \alpha(t)) + d(x_0, \beta(t)) - d(\alpha(t), \beta(t))].$$

The formula

$$d_{x_0}(a, b) = e^{-(a|b)_{x_0}}$$

defines an ultra-metric on ∂X.

4.2. **Conformal transformations and Radon-Nikodym derivatives.** The group $\mathrm{Aut}(X)$ of isometries of X acts on ∂X by conformal transformations. The Hausdorff dimension of $(\partial X, d_{x_0})$ equals $\log q$ and the normalized Hausdorff measure ν on $(\partial X, d_{x_0})$ is the unique Borel probability measure on ∂X invariant under the action of the stabilizer $K = \mathrm{Aut}(X)_{x_0}$ of x_0. The Radon-Nikodym derivative of $g \in \mathrm{Aut}(X)$ at $b \in \partial X$ is

$$\frac{dg_* \nu}{d\nu}(b) = q^{B_b(x_0, gx_0)}.$$

4.3. **The Koopman representation and the Harish-Chandra function.** The Koopman representation

$$\lambda_\nu : \mathrm{Aut}(X) \to U(L^2(\partial X, \nu))$$

is defined as

$$(\lambda_\nu(g) f)(b) = f(g^{-1} b) \sqrt{\frac{dg_* \nu}{d\nu}(b)}.$$

(The representation λ_ν is unitary equivalent to the quasi-regular representation $\lambda_{G/P}$, where $G = \mathrm{Aut}(X)$ and $P = G_b$, with $b \in \partial X$ any base point at infinity.)

Let $1_{\partial X} \in L^2(\partial X, \nu)$ be the constant function equal to 1. The Harish-Chandra function

$$\Xi : \mathrm{Aut}(X) \to (0, \infty)$$

is the coefficient of λ_ν defined by $1_{\partial X}$ that is:

$$\Xi(g) = \langle \lambda_\nu(g) 1_{\partial X}, 1_{\partial X} \rangle.$$

As the action of K preserves the measure and as λ_ν is unitary, the Harish-Chandra function is K-bi-invariant and symmetric. (In [23, Part II, 16] Harish-Chandra introduces the function Ξ on a connected reductive Lie group. The function Ξ can be viewed as the spherical function associated to a quasi-regular representation. The definitions make sense for G a locally compact (separable) unimodular group with a compact subgroup K such that (G, K) is a Gelfand pair, and an irreducible unitary representation π with one-dimensional K-invariant subspace [21, 1.5]. In our case $G = \mathrm{Aut}(X)$, $K = \mathrm{Aut}(X)_{x_0}$ and $\pi = \lambda_\nu$.)

4.4. **A formula for the Harish-Chandra function.** The length function

$$L : \mathrm{Aut}(X) \to \mathbb{N} \cup \{0\}$$

$$L(g) = d(x_0, gx_0)$$

is also K-bi-invariant and symmetric (notice that the elements of length 0 are the elements of K). As K acts transitively on each sphere of X with center x_0, if $g, g' \in \mathrm{Aut}(X)$ satisfy $L(g) = L(g')$ then there exist $k, k' \in K$ such that $kg = g'k'$. This implies that Ξ is constant on the level sets of L. For each $n \in \mathbb{N} \cup \{0\}$, we will write $\Xi(n)$ for the common value of the Harish-Chandra function on all $g \in \mathrm{Aut}(X)$ such that $L(g) = n$. We claim that for any $n \in \mathbb{N} \cup \{0\}$,

$$\Xi(n) = \left(1 + \frac{q-1}{q+1} n\right) q^{-n/2}. \tag{3}$$

If $L(g) = 0$, that is if $g \in K$, then $\Xi(g) = 1$. If $L(g) > 0$, let $[x_0, gx_0]$ be the (image of the) unique geodesic segment of X between x_0 and gx_0. For each vertex x of X at distance exactly 1 from $[x_0, gx_0]$, consider the ball U_x of ∂X of radius $e^{-d(x_0,x)}$ consisting of the points at infinity of the geodesic rays of X starting from x_0 and passing through x. We obtain the partition

$$\partial X = \bigcup_{d(x,[x_0,gx_0])=1} U_x.$$

As for each $r \in \mathbb{N}$ the measure of a ball of radius e^{-r} equals $[(q+1)q^{r-1}]^{-1}$, it is easy to check that

$$\begin{aligned}
\Xi(g) &= \int_{\partial X} q^{\frac{1}{2} B_b(x_0, gx_0)} d\nu(b) \\
&= \sum_{d(x,[x_0,gx_0])=1} \int_{U_x} q^{\frac{1}{2} B_b(x_0, gx_0)} d\nu(b) \\
&= \sum_{d(x,[x_0,gx_0])=1} \int_{U_x} q^{\frac{1}{2}(d(x_0,x) - d(gx_0,x))} d\nu(b) \\
&= \sum_{d(x,[x_0,gx_0])=1} q^{\frac{1}{2}(d(x_0,x) - d(gx_0,x))} \nu(U_x) \\
&= \left(1 + \frac{q-1}{q+1} n\right) q^{-n/2}.
\end{aligned}$$

4.5. **Computing operator norms with the Harish-Chandra function.** We first compute the norms of some operators defined by the Koopman representation. We then explain how spectral transfer applies to compute the norms of the corresponding operators defined by the regular representation.

Proposition 4.1. *Let $r \in \mathbb{N}$. Let Γ be the free group of rank r. Let $a_1, \cdots, a_r$ be a free generating set of Γ. Let X be the Cayley graph of Γ with respect to $\{a_1^{\pm 1}, \cdots, a_r^{\pm 1}\}$. Let $e = x_0 \in X$ be a base point. Let $G = Aut(X)$ and let $L(g) = d(x_0, gx_0)$ be the length function on G defined by x_0. For each integer $n \geq 0$, let $S_n = L^{-1}(n) \cap \Gamma$ and let $B_n = L^{-1}([0, n]) \cap \Gamma$ (where $\Gamma \subset G$ is the natural embedding). Then*

$$\|\lambda_\nu(\mu_{S_n})\| = \Xi(n),$$

$$\|\lambda_\nu(\mu_{B_n})\|=\frac{1}{|B_n|}\sum_{k=0}^{n}\Xi(k)|S_k|.$$

Proof. We first consider the case of the sphere S_n. Let $1_{\partial X}$ be the constant function equal to 1 on ∂X. Applying the Cauchy-Schwarz inequality and the fact that the function Ξ is constant on S_n, we obtain:

$$\begin{aligned}\|\lambda_\nu(\mu_{S_n})\| &\geq \|\lambda_\nu(\mu_{S_n})1_{\partial X}\|_2 = \|\lambda_\nu(\mu_{S_n})1_{\partial X}\|_2\|1_{\partial X}\|_2\\ &\geq \langle\lambda_\nu(\mu_{S_n})1_{\partial X},1_{\partial X}\rangle = \frac{1}{|S_n|}\sum_{\gamma\in S_n}\langle\lambda_\nu(\gamma)1_{\partial X},1_{\partial X}\rangle\\ &= \Xi(n).\end{aligned}$$

To prove the other inequality, we first notice that for $p\in\{1,2,\infty\}$, the operator $\lambda_\nu(\mu_{S_n}) : L^p(X,\nu)\to L^p(X,\nu)$ is bounded. In the case $p=2$ it is self-adjoint. Thanks to Riesz-Thorin's theorem,

$$\|\lambda_\nu(\mu_{S_n})\|_{2\to 2}\leq \|\lambda_\nu(\mu_{S_n})\|_{\infty\to\infty}.$$

As $\lambda_\nu(\mu_{S_n})$ preserves positive functions, it is obvious that

$$\|\lambda_\nu(\mu_{S_n})\|_{\infty\to\infty}=\|\lambda_\nu(\mu_{S_n})1_{\partial X}\|_\infty.$$

We claim that the function $\lambda_\nu(\mu_{S_n})1_{\partial X}$ is constant equal to $\Xi(n)$. To prove the claim we first show that the function is K-invariant (hence constant as K acts transitively on ∂X). Let $b\in\partial X$ and $k\in K$. We have:

$$\begin{aligned}\lambda_\nu(\mu_{S_n})1_{\partial X}(kb) &= \frac{1}{|S_n|}\sum_{\gamma\in S_n}q^{\frac{1}{2}B_{kb}(x_0,\gamma x_0)}\\ &= \frac{1}{|S_n|}\sum_{\gamma\in S_n}q^{\frac{1}{2}B_{b}(k^{-1}x_0,k^{-1}\gamma x_0)}\\ &= \frac{1}{|S_n|}\sum_{\gamma\in S_n}q^{\frac{1}{2}B_{b}(x_0,k^{-1}\gamma x_0)}\\ &= \frac{1}{|S_n|}\sum_{\gamma\in S_n}q^{\frac{1}{2}B_{b}(x_0,\gamma x_0)}\\ &= \lambda_\nu(\mu_{S_n})1_{\partial X}(b).\end{aligned}$$

The claim is proved because ν is a probability measure and by definition of Ξ we have:

$$\int_{\partial X}\lambda_\nu(\mu_{S_n})1_{\partial X}(b)d\nu(b)=\Xi(n).$$

In the he case of the ball B_n, applying Cauchy-Schwarz's inequality and Riesz-Thorin's theorem in similar ways proves that

$$\|\lambda_\nu(\mathbf{1}_{B_n})\|=\sum_{\gamma\in B_n}\Xi(\gamma).$$

The result follows because Ξ is constant on spheres. □

Proposition 4.2. *Let $r\in\mathbb{N}$. Let Γ be the free group of rank r. Let $a_1,\cdots,a_r$ be a free generating set of Γ. Let $S=\{a_1^{\pm1},\cdots,a_r^{\pm1}\}$. For each integer $n\geq 0$, let S_n,* *respectively B_n, be the sphere, respectively the ball, around $e \in \Gamma$ of radius n with respect to the word metric on Γ defined by S. Let $q = 2r - 1$. Then*

$$\|\rho_\Gamma(\mu_{S_n})\| = \left(1 + \frac{q-1}{q+1} n\right) q^{-n/2},$$

$$\|\rho_\Gamma(\mu_{B_n})\| = c(q,n) \left(1 + \left(1 + \frac{1}{\sqrt{q}}\right) n\right) q^{-n/2},$$

where $c(q,n) = \left(1 + 2q^{-n} \sum_{k=0}^{n-1} q^k\right)^{-1}$.

Proof. We claim that for any positive element $m \in \mathbb{C}[\Gamma]$, we have

$$\|\rho_\Gamma(m)\| = \|\lambda_\nu(m)\|.$$

The inequality $\|\rho_\Gamma(m)\| \leq \|\lambda_\nu(m)\|$ follows from [36, Lemma 2.3] because $1_{\partial X}$ is a positive vector for λ_ν. The inequality $\|\rho_\Gamma(m)\| \geq \|\lambda_\nu(m)\|$ is true for any element $m \in \mathbb{C}[\Gamma]$ (not only positive ones), because the action of Γ on ∂X is amenable, see [26] and [16, Theorem 7]. To prove the proposition, we apply Proposition 4.1 and Formula 3. In the case of the sphere S_n, this immediately proves the statement. The case of the ball B_n requires some computation. If $q = 1$, the formula is obvious. If $q > 1$, we have

$$\begin{aligned}
\|\rho_\Gamma(\mu_{B_n})\| &= \frac{1}{|B_n|} \sum_{k=0}^{n} \Xi(k) |S_k| \\
&= \frac{1}{|B_n|} \left(1 + \frac{q + q^{1/2}}{q} n\right) q^{n/2} \\
&= \left(\frac{q+1}{q-1} q^n - \frac{2}{q-1}\right)^{-1} \left(1 + \frac{q + q^{1/2}}{q} n\right) q^{n/2} \\
&= c(q,n) \left(1 + \left(1 + \frac{1}{\sqrt{q}}\right) n\right) q^{-n/2},
\end{aligned}$$

where $c(q,n) = \left(1 + 2q^{-n} \sum_{k=0}^{n-1} q^k\right)^{-1}$. □

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